Improvement in Model Reduction Schemes Using the System Equivalent Reduction Expansion Process

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Introduction

MOST system modeling is performed with the finite element (FE) method. The model consists of mass and stiffness matrices, which are useful in static analyses, load predictions, and/or dynamic response calculations. Unfortunately, an FE analysis produces large matrices, which often leads to expensive eigenanalyses. A model reduction will, therefore, reduce the system to a manageable size. Furthermore, a reduced model is necessary for test-analysis correlation studies in modal testing.

The first reduction method is that due to Guyan[1] and Irons.[2] They partition the mass and stiffness matrices into a set of master and slave degrees of freedom (DOF). A coordinate transformation using the partitioned stiffness submatrices then reduces the model to the master DOF. Although this is the most common reduction scheme, it only approximates the eigenanalysis of the full system, and the results depend upon the type and number of master DOF. Typically, though, the lower system modes will be accurate, whereas the higher modes can be severely in error.

Attempts to improve the Guyan results[1] include using static shape functions from unit loads applied to the master DOF in the transformation matrix[3] and statically taking into account the missing inertia forces at the slave DOF.[4] Although these methods show improvement over the standard Guyan technique, they are still approximate. The modal reduction methods of O'Callahan et al.[5] and Kammer,[6] however, are exact reduction techniques that recover the selected modes in a full system eigenanalysis. This paper will make use of the System Equivalent Reduction Expansion Process (SEREP) as defined by O'Callahan et al.[5] Although this method is exact, it is inefficient because it requires the full system eigenvectors for the set of modes of interest. An alternate approach using the eigenvectors from any reduced model is therefore proposed to avoid using the full system eigenvectors.

Problem Formulation

Specifying a displacement set of $r \times 1$ master (x_m) and $(n-r) \times 1$ slave (x_s) DOF and partitioning the mass and stiffness matrices as

$$M = \begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix}_{n \times n} \qquad K = \begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix}_{n \times n} \tag{1}$$

result in the reduced homogeneous dynamic system

$$\tilde{M}\ddot{x}_m + \tilde{K}x_m = 0 \tag{2}$$

where

$$\tilde{M} = P^T M P$$

$$\tilde{K} = P^T K P$$

and P defines the $n \times r$ transformation matrix between the full set x and master set x_m. Each reduction scheme can then be identified from the structure of matrix P.

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The classical static reduction is given by the Guyan (Irons) transformation[1,2]

$$P_G = \begin{bmatrix} I \\ -K_{ss}^{-1}K_{sm} \end{bmatrix} \tag{3}$$

where I is an $r \times r$ identity matrix. Unfortunately, the transformation P_G is only approximate because it assumes negligible inertia forces at the slave DOF. The Improved Reduced System (IRS)[4] method, however, defines the transformation as

$$P_{\mathrm{IRS}} = P_G + SMP_G\tilde{M}_G^{-1}\tilde{K}_G \tag{4}$$

where

$$S = \begin{bmatrix} 0 & 0 \\ 0 & K_{ss}^{-1} \end{bmatrix}$$

and $\tilde{M}_G$ and $\tilde{K}_G$ are the Guyan reduced mass and stiffness matrices,[1] respectively. Although the IRS transformation is an improvement, it is still approximate (albeit a better one). The SEREP method,[5] on the other hand, preserves the natural frequencies and mode shapes during the reduction process. This method begins by computing the eigenvectors of the full system and arranging them into an $n \times n$ modal matrix

$$\Phi = [\phi_1 \quad \phi_2 \quad \cdots \quad \phi_n] \equiv \begin{bmatrix} \Phi_m \\ \Phi_s \end{bmatrix} \tag{5}$$

where $\phi_i, i = 1, \ldots, n$ represent the individual eigenvectors and Φ_m and Φ_s denote the $r \times n$ and $(n-r) \times n$ modal submatrices corresponding to the master and slave DOF partitions of Φ, respectively. Retaining only r modes of interest, the SEREP transformation is given as

$$P_{\mathrm{SEREP}} = \begin{bmatrix} \bar{\Phi}_m \\ \bar{\Phi}_s \end{bmatrix} \bar{\Phi}_m^{-1} \tag{6}$$

where $\bar{\Phi}_m$ represents an $r \times r$ submatrix of desired modes from the rectangular matrix Φ_m in Eq. (5).

The SEREP transformation is then given by Eq. (6) and only requires the availability of the full eigenvectors. This limitation can be effectively removed by the following considerations. First, perform any reduction scheme (e.g., Guyan[1]) to obtain the $r \times r$ mass and stiffness matrices. An eigenanalysis on these reduced matrices yields the natural frequencies and mode shapes and is given by

$$\begin{aligned} \Omega^2_{r \times r} &= \mathrm{diag}\left(\omega_1^2 \quad \cdots \quad \omega_r^2\right) \\ \Phi_{r \times r} &= \left[\phi_{m_1} \quad \cdots \quad \phi_{m_r}\right] \end{aligned} \tag{7}$$

The modal matrix Φ in Eq. (7) is then dynamically expanded[7] to the slave DOF from

$$\phi_{s_i} = -\left(K_{ss} - \omega_i^2 M_{ss}\right)^{-1}\left(K_{sm} - \omega_i^2 M_{sm}\right)\phi_{m_i}, \qquad i = 1, \ldots, r \tag{8}$$

where M_{ss}, M_{sm}, K_{ss}, and K_{sm} are given in Eq. (1) and ω_i^2 and ϕ_{m_i} are given in Eq. (7). The expanded modal matrix can now be formed by

$$\Phi = \begin{bmatrix} \phi_{m_1} & \cdots & \phi_{m_r} \\ \phi_{s_1} & & \phi_{s_r} \end{bmatrix} \equiv [\phi_1 \quad \cdots \quad \phi_r] \tag{9}$$

Although, in principle, Eq. (8) could be used, it is often likely that the required inversion is costly. Miller's expansion[8] is then used to circumvent the needed inversion for every reduced frequency:

$$\phi_{s_i} = -\left(K_{ss}^{-1}K_{sm} + \omega_i^2\left[-K_{ss}^{-1}M_{sm} + K_{ss}^{-1}M_{ss}K_{ss}^{-1}K_{sm}\right]\right)\phi_{m_i} \qquad i = 1, \ldots, r \tag{10}$$

or the more exact Kidder expansion[7]

$$\phi_{s_i} = -\left(K_{ss}^{-1} + \omega_i^2 K_{ss}^{-1}M_{ss}K_{ss}^{-1}\right)\left(K_{sm} - \omega_i^2 M_{sm}\right)\phi_{m_i} \qquad i = 1, \ldots, r \tag{11}$$

could be used.

To satisfy mass orthonormality, it is proposed to use Baruch and Itzhack's method.[9] That is, normalize Φ by

$$\bar{\phi}_i = \phi_i \left(\phi_i^T M \phi_i\right)^{-\frac{1}{2}}, \qquad i = 1, \ldots, r \tag{12}$$

where ϕ_i is the ith mode shape before normalization and given in Eq. (9) and M is as in Eq. (1). The expanded, mass normalized modal matrix is then finally computed from

$$X = \bar{\Phi}(\bar{\Phi}^T M \bar{\Phi})^{-\frac{1}{2}} \equiv \begin{bmatrix} X_m \\ X_s \end{bmatrix} \tag{13}$$

where $\bar{\Phi}$ is the modal matrix formed from the r normalized mode shapes in Eq. (12). The new SEREP transformation now becomes

$$P_{\text{SEREP}} = \begin{bmatrix} X_m \\ X_s \end{bmatrix} X_m^{-1} \tag{14}$$

If Guyan reduction[1] is used in Eq. (14), the method can be called SEREP-Guyan. Alternatively, if the IRS method is used, the method can be called SEREP-IRS. The particular reduction scheme, however, is unimportant. The only underlying assumption is that the shape of each reduced mode resembles the full system mode, which inevitably is dependent on the choice of master DOF. A number of authors have considered the selection of an appropriate master set.[10–12] Suffice it to say, good results are expected through a sufficient number and location of master DOF.

Results

A numerical example of a uniform cantilevered beam is used to validate the proposed reduction scheme. The beam was modeled as steel with material and geometric properties listed in Table 1. It was discretized into 12 beam two-DOF/node finite elements (transverse displacement and rotation). A consistent mass approach was used to formulate the mass matrix. The full system matrices were then of size 24×24. Node 1 is located 0.05 m from the wall and node 2 is 0.05 m from node 1. The other nodes are equidistant from each other at 0.045 m.

The mass and stiffness matrices were then partitioned and reduced to the transverse DOF at nodes 2, 4, 6, 8, 10, and 12, resulting in a 6×6 reduced model. These nodal locations represent a good set of master DOF. The natural frequencies from the different reduction methods are shown in Table 2. Compared with the exact frequencies, Guyan[1] results in larger errors as the frequency increases. This is expected since the neglected inertia terms dominate for the higher modes. The SEREP-Guyan method, however, has compensated for the error and yields almost exact frequencies up to the fifth mode (the sixth being only approximate). For comparison purposes, the results from the IRS method are presented. Although this method does exceptionally well in identifying all of the modes, the fifth and sixth modes are still somewhat in error. This frequency error has been corrected in the SEREP-IRS method. This method uses the IRS reduced mass and stiffness matrices instead of the Guyan matrices. The accuracy of the various reduced mode shapes to the exact SEREP reduced mode shapes is obtained from a MAC[13] correlation and is shown in Table 3. The MAC was evaluated only for modes 4, 5, and 6 because these modes exhibited the largest frequency error. Observe that the SEREP-Guyan mode shapes correspond more closely to the exact mode shapes than those of Guyan.[1] A similar trend is shown for SEREP-IRS.

Table 1 Material and geometric properties of beam

Property	E, Pa	ρ, kg/m^3	A, m^2	L, m	I, m^4
Value	2.07×10^{11}	8130	1.5×10^{-4}	0.55	4.5×10^{-10}

Table 2 Beam natural frequencies (hertz) from different reduction methods

Mode	Exact	Guyan	SEREP-Guyan[a]	IRS	SEREP-IRS[a]
1	16.1676	16.1678	16.1676	16.1676	16.1676
2	101.3223	101.3645	101.3223	101.3223	101.3223
3	283.7346	284.6959	283.7347	283.7347	283.7346
4	556.1992	564.0555	556.2121	556.2097	556.1992
5	920.1960	955.0876	920.5521	921.0429	920.1977
6	1376.7569	1433.1140	1384.8284	1383.0592	1376.9494

[a]Using exact dynamic expansion.

Table 3 MAC between SEREP and the varying reduction methods

Mode	Guyan	SEREP-Guyan[a]	IRS	SEREP-IRS[a]
4	0.9998	1.0000	1.0000	1.0000
5	0.9997	1.0000	1.0000	1.0000
6	0.9911	0.9951	0.9997	0.9999

[a]Using exact dynamic expansion.

Table 4 Beam natural frequencies (hertz) using different dynamic expansions in SEREP-Guyan

Mode	Exact expansion	Miller expansion	Kidder expansion
1	16.1676	16.1676	16.1676
2	101.3223	101.3223	101.3223
3	283.7347	283.7347	283.7347
4	556.2121	556.2097	556.2032
5	920.5521	921.0429	920.7523
6	1384.8284	1383.0592	1382.8326

The preceding results show that SEREP-Guyan and SEREP-IRS are a considerable improvement over the standard Guyan reduction.[1] However, the exact expansion relationship given in Eq. (8) was used. Table 4 shows the reduced frequencies from SEREP-Guyan using the approximate dynamic expansions of Miller[8] and Kidder.[7] It is seen that both approximate expansions also yield excellent frequencies as compared with the exact expansion frequencies. The simple cantilevered beam example clearly shows that an explicit inversion is not strictly required when dynamically expanding the reduced modes. Whether this is true for complex structures needs further investigation.

Conclusions

The SEREP produces an exact reduced model. However, this method requires knowledge of the full system eigenvectors. An extension of this method was presented that uses only the mode shapes of a reduced system. The reduced system mode shapes can be obtained from any reduction procedure, including Guyan, for example. Results from a simulation of a cantilevered Euler–Bernoulli beam validated the technique. It was shown that the method improved upon the Guyan and the IRS reduction methods provided that a proper set of master DOF were chosen.

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